

# Dealing With 4-Variables by Resolution: An Improved MaxSAT Algorithm\*

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## Abstract

We study techniques for solving the MAXIMUM SATISFIABILITY problem (MAXSAT). Our focus is on variables of degree 4. We identify cases for degree-4 variables and show how the resolution principle and the kernelization techniques can be nicely integrated to achieve more efficient algorithms for the MAXSAT problem. As a result, we present an algorithm of time  $O^*(1.3248^k)$  for the MAXSAT problem, improving the previous best upper bound  $O^*(1.358^k)$  by Ivan Bliznets and Alexander.

## 1 Introduction

The SATISFIABILITY problem (SAT) is of fundamental importance in computer science and information technology [3]. Its optimization version, the MAXIMUM SATISFIABILITY problem (MAXSAT) plays a similar role in the study of computational optimization, in particular in the study of approximation algorithms [12]. Since the problems are NP-hard [8], different algorithmic approaches, including heuristic algorithms (e.g., [9, 18]), approximation algorithms (e.g., [1, 22]), and exact and parameterized algorithms (e.g., [4, 5, 17]), have been extensively studied.

The main result of the current paper is an improved parameterized algorithm for the MAXSAT problem. Formally, the (parameterized) MAXSAT problem is defined as follows.<sup>1</sup>

MAXSAT: Given a CNF formula  $F$  and an integer  $k$  (the *parameter*), is there an assignment to the variables in  $F$  that satisfies at least  $k$  clauses in  $F$ ?

It is known that the MAXSAT problem is fixed-parameter tractable, i.e., it is solvable in time  $O^*(f(k))$  for a function  $f$  that only depends on the parameter  $k$ .<sup>2</sup> The research on parameterized algorithms for the MAXSAT problem has been focused on improving the upper bound on the function  $f$ , with an impressive list of improvements. The table in Figure 1 summarizes the progress in the research. For comparison, we have also included our result in the current paper in the table.

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<sup>1</sup>We remark that there is a variation of the MAXSAT problem, that asks whether there is an assignment to satisfy at least  $k + m/2$  clauses in a CNF formula with  $m$  clauses [15]. This variation has also drawn significant attention.

<sup>2</sup>Following the current convention in the research in exact and parameterized algorithms, we will use the notation  $O^*(f(k))$  to denote the bound  $f(k)n^{O(1)}$ , where  $n$  is the instance size.

Bound	Reference	Year
$O^*(1.618^k)$	Mahajan, Raman [15]	1999
$O^*(1.400^k)$	Niedermeier, Rossmanith [16]	2000
$O^*(1.381^k)$	Bansal, Raman [2]	1999
$O^*(1.370^k)$	Chen, Kanj [5]	2002
$O^*(1.358^k)$	Bliznets, Golovnev [4]	2012
$O^*(1.325^k)$	this paper	2014

Figure 1: Progress in MAXSAT algorithms

Most algorithms for SAT and MAXSAT are based on the branch-and-bound process [9]. The *Strong Exponential Time Hypothesis* conjectures that SAT cannot be solved in time  $O^*(2^{cn})$  for any constant  $c < 1$ , where  $n$  is the number of variables in the input CNF formula [13]. The hypothesis indicates, to some extent, a popular opinion that branch-and-bound is perhaps unavoidable in solving the SAT problem and its variations.

Therefore, how to branch more effectively in algorithms solving SAT and MAXSAT has become critical. In particular, all existing parameterized algorithms for MAXSAT and most known algorithms for SAT have been focused on more effective branching strategies to further improve the algorithm complexity. Define the *degree* of a variable  $x$  in a CNF formula  $F$  to be the number of times  $x$  and  $\bar{x}$  appear in the formula. For MAXSAT, it is well-known that branching on variables of large degree will be sufficiently effective. On the other hand, variables of degree bounded by 2 can be handled efficiently based on the resolution principle [7]. Recently, Bliznets and Golovnev [4] proposed new strategies for branching on degree-3 variables more effectively and improved Chen and Kanj’s algorithm [5], which had stood as the best algorithm for MAXSAT for 10 years.

For further improving the algorithm complexity for MAXSAT, the next bottleneck is on degree-4 variables. Degree-4 variables seem neither to have a large enough degree to support direct branchings of sufficient efficiency, nor to have simple enough structures that allow feasible case-by-case analysis to yield more efficient manipulations. In fact, degree-4 variables are the sources for the worst branching cases in Chen-Kanj’s algorithm (case 3.10 in [5]) as well as in Bliznets-Golovnev’s algorithm (Theorem 5, step 10 in [4]).

A contribution of the current paper is to show how the resolution principle [7] can be used in handling degree-4 variables in solving the MAXSAT problem. It has been well-known that the resolution principle is a very powerful tool in solving the SAT problem [7]. In particular, variable resolutions in a CNF formula preserve the satisfiability of the formula. Unfortunately, variable resolutions cannot be used directly in solving the MAXSAT problem in general case because they do not provide a predictable decreasing in the maximum number of clauses in the CNF formulas that can be satisfied by an assignment. In particular, for a degree-4 variable  $x$  in a CNF formula  $F$  for which an optimal assignment satisfies  $k$  clauses, the resolution on  $x$  may result in CNF formulas for which optimal assignments satisfy  $k - 4$ ,  $k - 2$ ,  $k - 1$ , and  $k$  clauses, respectively.

We identify cases for degree-4 variables and show how the resolution principle can be applied efficiently on these cases (see our reduction rules R-Rules 6-7). This technique helps us to eliminate the structures that do not support efficient branchings. We also show how the resolution principle and kernelization algorithms of parameterized problems are nicely integrated. Note that resolutions may significantly increase the size and the number of clauses in a formula. However, it turns out to be not a concern for algorithms for MAXSAT: MAXSAT has a polynomial-time kernelization algorithm [5] that can bound the size of the formula  $F$  by  $O(k^2)$  in an instance  $(F, k)$  of MAXSAT. Therefore, the resolution principle can be used whenever it is applicable – once the formula size gets too large, we can simply use the kernelization algorithm to reduce the formula size. In fact,

one of our reduction rules (R-Rule 7) does not even decrease the parameter value, which, however, can only be applied polynomial many times because of the kernelization of MAXSAT.

A nice approach suggested by Bliznets and Golovnev [4] is to transform solving MAXSAT on a class of special instances into solving the SET-COVER problem. However, the method proposed in [4] is not efficient enough to achieve our bound. For this, we introduce new branching rules that are sufficiently efficient and further reduce the instances to an even more restricted form. In particular, we show how to eliminate all clauses of size bounded by 3. The restricted form of the instances allow us to apply more powerful techniques in randomized algorithms and in derandomization [22] to derive tighter lower bounds on the instances of MAXSAT, which makes it become possible to use more effectively the existing algorithm for SET-COVER [19].

The paper is organized as follows. Section 2 provides preliminaries and necessary definitions. Section 3 describes our reduction rules, which are the polynomial-time processes that can be used to simplify the problem instances. Branching rules are given in Section 4 that are applied on instances of specified structures. A complete algorithm is presented in Section 5. Conclusions and remarks are given in Section 6 where we also discuss the difficulties of further improving the results presented in the current paper.

## 2 Preliminary

A (Boolean) *variable*  $x$  can be assigned value either 1 (TRUE) or 0 (FALSE). A variable  $x$  has two corresponding literals: the *positive literal*  $x$  and the *negative literal*  $\bar{x}$ , which will be called the *literals* of  $x$ . The variable  $x$  is called the *variable for the literals*  $x$  and  $\bar{x}$ . A *clause*  $C$  is a disjunction of a set of literals, which can be regarded as a set of the literals. Hence, we may write  $C_1 = zC_2$  to indicate that the clause  $C_1$  consists of the literal  $z$  plus all literals in the clause  $C_2$ , and use  $C_1C_2$  to denote the clause that consists of all literals that are in either  $C_1$  or  $C_2$ , or both. Without loss of generality, we assume that a literal can appear in a clause at most once. A clause  $C$  is *satisfied* by an assignment if under the assignment, at least one literal in  $C$  gets a value 1. A (CNF Boolean) *formula*  $F$  is a conjunction of clauses  $C_1, \dots, C_m$ , which can be regarded as a collection of the clauses. The formula  $F$  is *satisfied* by an assignment to the variables in the formula if all clauses in  $F$  are satisfied by the assignment. Throughout this paper, denote by  $n$  the number of variables and by  $m$  the number of clauses in a formula.

A literal  $z$  is an  $(i, j)$ -*literal* in a formula  $F$  if  $z$  appears  $i$  times and  $\bar{z}$  appears  $j$  times in the formula  $F$ . A variable  $x$  is an  $(i, j)$ -*variable* if the positive literal  $x$  is an  $(i, j)$ -literal. Therefore, a variable  $x$  has degree  $h$  if  $x$  is an  $(i, j)$ -variable such that  $h = i + j$ . A variable of degree  $h$  is also called an  $h$ -*variable*. A variable is an  $h^+$ -*variable* if its degree is at least  $h$ .

The *size* of a clause  $C$  is the number of literals in  $C$ . A clause is an  $h$ -*clause* if its size is  $h$ , and an  $h^+$ -*clause* if its size is at least  $h$ . A clause is *unit* if its size is 1 and is *non-unit* if its size is larger than 1. The *size* of a CNF formula  $F$  is equal to the sum of the sizes of the clauses in  $F$ .

A *resolvent* on a variable  $x$  in a formula  $F$  is a clause of the form  $CD$  such that  $xC$  and  $\bar{x}D$  are clauses in  $F$ . The *resolution* on the variable  $x$  in  $F$  is the conjunction of all resolvents on  $x$ .

An instance  $(F, k)$  of the MAXSAT problem asks whether there is an assignment to the variables in a given CNF formula  $F$  that satisfies at least  $k$  clauses in  $F$ . It is known [5] that with a simple polynomial-time preprocessing (i.e. a *kernelization algorithm*), we can assume an  $O(k^2)$  upper bound on the size of the formula  $F$ . The kernelization algorithm proceeds as follows (see [5] for a detailed discussion): (1) if the number of clauses in  $F$  is at least  $2k$ , then  $(F, k)$  is a Yes-instance; and (2) if a clause  $C$  in  $F$  has size at least  $k$ , then we can instead work on the instance  $(F \setminus \{C\}, k - 1)$ . After this polynomial-time preprocessing, we can assume that in the instance  $(F, k)$ , the formula

$F$  contains at most  $2k$  clauses and each clause in  $F$  has its size bounded by  $k - 1$ . This implies that the size of the formula  $F$  is bounded by  $2k(k - 1) = O(k^2)$ .

In a typical branch-and-bound algorithm for the MAXSAT problem, a branching step on an instance  $(F, k)$  of MAXSAT produces, in polynomial time, a collection  $\{(F_1, k - d_1), \dots, (F_r, k - d_r)\}$  of instances of MAXSAT, where each  $d_i$  is a positive integer bounded by  $k$ , such that  $(F, k)$  is a Yes-instance if and only if at least one of  $(F_1, k - d_1), \dots, (F_r, k - d_r)$  is a Yes-instance. Such a branching step is called a  $(d_1, \dots, d_r)$ -branching, the vector  $t = (d_1, \dots, d_r)$  is called the *branching vector* for the branching, and each instance  $(F_i, k - d_i)$ ,  $1 \leq i \leq r$ , is called a *branch* of the branching. It can be shown (see, e.g. [6]) that the polynomial

$$p_t(x) = x^k - x^{k-d_1} - \dots - x^{k-d_r} = x^{k-d_{\max}}(x^{d_{\max}} - x^{d_{\max}-d_1} - \dots - x^{d_{\max}-d_r}),$$

where  $d_{\max} = \max\{d_1, \dots, d_r\}$ , has a unique positive root that is larger than or equal to 1. For the branching vector  $t = (d_1, \dots, d_r)$ , denote this positive root of  $p_t(x)$  by  $\rho(t)$ , and call it the *branching complexity* of the branching.

Let  $t_1$  and  $t_2$  be two branching vectors. We say that the  $t_1$ -branching is *inferior* to the  $t_2$ -branching if the branching complexity of the  $t_1$ -branching is larger than that of the  $t_2$ -branching, i.e., if  $\rho(t_1) > \rho(t_2)$ . It is not difficult to verify the following facts:

- for two branching vectors  $(d_1, \dots, d_r)$  and  $(d'_1, \dots, d'_r)$ , such that  $d_i \leq d'_i$  for all  $i$  and that  $d_j < d'_j$  for at least one  $j$ , the  $(d_1, \dots, d_r)$ -branching is inferior to the  $(d'_1, \dots, d'_r)$ -branching. That is, smaller decreasing in the parameter leads to a higher branching complexity; and
- for two branching vectors  $(d_1, d_2)$  and  $(d'_1, d'_2)$ , such that  $d_1 + d_2 = d'_1 + d'_2$  and  $\max\{d_1, d_2\} > \max\{d'_1, d'_2\}$ , the  $(d_1, d_2)$ -branching is inferior to the  $(d'_1, d'_2)$ -branching. That is, a less “balanced” branching has a higher branching complexity.

It is well-known in parameterized algorithms that for a parameterized algorithm based on the branch-and-bound process, if every branching step in the algorithm has its branching complexity bounded by a constant  $c \geq 1$ , then the running time of the algorithm is bounded by  $O^*(c^k)$ .

### 3 Reduction rules

A *reduction rule* transforms, in polynomial time, an instance  $(F, k)$  of MAXSAT into another instance  $(F', k')$  with  $k \geq k'$  such that  $(F, k)$  is a Yes-instance if and only if  $(F', k')$  is a Yes-instance. Note that a reduction rule can be regarded as a special case of branching steps that on an instance  $(F, k)$  produces a single instance  $(F', k')$ . If  $k > k'$ , then this branching has a branching complexity 1, which is the best possible and is not inferior to any other kind of branchings.

We present seven reduction rules, R-Rules 1-7. The reduction rules are supposed to be applied *in order*, i.e., R-Rule  $j$  is applied only when none of R-Rules  $i$  with  $i < j$  is applicable. In the following,  $F$  is always supposed to be a conjunction of clauses.

The first three reduction rules are from [5].

**R-Rule 1.**  $(F \wedge (x\bar{x}C), k) \rightarrow (F, k - 1)$ , and  $(F \wedge (x) \wedge (\bar{x}), k) \rightarrow (F, k - 1)$ .

**R-Rule 2.** If there is an  $(i, j)$ -literal  $z$  in the CNF formula  $F$ , with at least  $j$  unit clauses  $(z)$ , then  $(F, k) \rightarrow (F_{z=1}, k - i)$ , where  $F_{z=1}$  is the formula  $F$  with an assignment  $z = 1$  on the literal  $z$ .

Assume that R-Rule 2 is not applicable to  $F$ , then  $F$  has no *pure literals*, i.e., literals whose negation does not appear in  $F$ . Thus, all variables are  $2^+$ -variables. Under this condition, we can process 2-variables based on the resolution principle [7], whose correctness can be easily verified.

**R-Rule 3.** For a 2-variable  $x$ ,  $(F \wedge (xC_1) \wedge (\bar{x}C_2), k) \rightarrow (F \wedge (C_1C_2), k - 1)$ .

In case none of R-Rules 1-3 is applicable, every variable is a  $3^+$ -variable. Moreover, for each  $(i, 1)$ -literal  $z$ , there is no unit clause  $(z)$ , and for each  $(i, 2)$ -literal  $z$ , there is at most one unit clause  $(z)$ . Now we describe two reduction rules from [4] (Simplification Rule 5 and Corollary 1 in [4]), which are based on variations of the resolution principle.

**R-Rule 4 ([4]).** For a  $(2, 1)$ -literal  $z$  and an arbitrary literal  $y$ ,  $(F \wedge (zy) \wedge (zC_2) \wedge (\bar{z}C_3), k) \rightarrow (F \wedge (yC_3) \wedge (\bar{y}C_2C_3), k - 1)$ .

**R-Rule 5 ([4]).** For a CNF formula  $F_0 = F \wedge (zC_1) \wedge (zC_2) \wedge (\bar{z}C_3)$ , where  $z$  is a  $(2, 1)$ -literal in  $F_0$  and  $C_1 \cup C_2 \cup C_3$  contains both  $y$  and  $\bar{y}$  for some variable  $y$ ,  $(F \wedge (zC_1) \wedge (zC_2) \wedge (\bar{z}C_3), k) \rightarrow (F \wedge (C_1C_3) \wedge (C_2C_3), k - 1)$ .

Therefore, in case none of R-Rules 1-5 is applicable, for each  $(2, 1)$ -literal  $z$ , the two clauses containing  $z$  are  $3^+$ -clauses. Now, we introduce two new reduction rules that are based on the resolution principle.

**R-Rule 6.** For an  $(i, 1)$ -literal  $z$  in a formula  $F_1 = F \wedge (zC_1) \wedge \dots \wedge (zC_i) \wedge (\bar{z}yC)$ , where  $y$  is a  $(j, 1)$ -literal for some  $j$ ,  $(F \wedge (zC_1) \wedge \dots \wedge (zC_i) \wedge (\bar{z}yC), k) \rightarrow (F \wedge (yCC_1) \wedge \dots \wedge (yCC_i), k - 1)$ .

**Lemma 3.1** *R-Rule 6 is safe, i.e., R-Rule 6 transforms the instance  $(F_1, k)$  of MAXSAT into an instance  $(F_2, k - 1)$  such that  $(F_1, k)$  is a Yes-instance if and only if  $(F_2, k - 1)$  is a Yes-instance.*

PROOF. First note that by the assumption, the formula  $F$  contains neither  $z$  nor  $\bar{z}$ .

Suppose that  $(F \wedge (zC_1) \wedge \dots \wedge (zC_i) \wedge (\bar{z}yC), k)$  is a Yes-instance with an optimal assignment  $\sigma_1$  that satisfies at least  $k$  clauses in  $F \wedge (zC_1) \wedge \dots \wedge (zC_i) \wedge (\bar{z}yC)$ . If  $\sigma_1$  does not satisfy  $(zC_d)$  for some  $d$ , then  $\sigma_1(z) = 0$ . Since  $z$  is an  $(i, 1)$ -literal,  $\bar{z}$  appears only in the clause  $(\bar{z}yC)$ . Therefore, if we re-assign  $z = 1$ , then the clause  $(zC_d)$  becomes satisfied, and the only clause that is satisfied by  $\sigma_1$  now may become unsatisfied is  $(\bar{z}yC)$ . Thus, the re-assignment  $z = 1$  in  $\sigma_1$  gives another optimal assignment  $\sigma'_1$  that satisfies all  $(zC_g)$ , for  $1 \leq g \leq i$ . Now if  $\sigma'_1$  does not satisfy  $(\bar{z}yC)$ , then we re-assign  $y = 1$ , which, because  $y$  is a  $(j, 1)$ -literal, gives a third optimal assignment  $\sigma''_1$  that satisfies all  $i + 1$  clauses in  $(zC_1) \wedge \dots \wedge (zC_i) \wedge (\bar{z}yC)$  (note that  $y$  cannot be  $\bar{z}$ ). Moreover,  $\sigma''_1$  satisfies at least  $k - (i + 1)$  clauses in  $F$ . By the resolution principle, the assignment  $\sigma''_1$  also satisfies all  $i$  clauses in  $(yCC_1) \wedge \dots \wedge (yCC_i)$ . Thus,  $\sigma''_1$  satisfies at least  $k - (i + 1) + i = k - 1$  clauses in  $F \wedge (yCC_1) \wedge \dots \wedge (yCC_i)$ , i.e.,  $(F \wedge (yCC_1) \wedge \dots \wedge (yCC_i), k - 1)$  is a Yes-instance.

Consider the other direction, suppose that  $(F \wedge (yCC_1) \wedge \dots \wedge (yCC_i), k - 1)$  is a Yes-instance with an optimal assignment  $\sigma_2$  that satisfies at least  $k - 1$  clauses in  $F \wedge (yCC_1) \wedge \dots \wedge (yCC_i)$ . First note that  $y$  is still a  $(j', 1)$ -literal in the formula  $F \wedge (yCC_1) \wedge \dots \wedge (yCC_i)$  for some  $j'$  (though  $j'$  may not be  $j$ ). Therefore, if  $\sigma_2$  does not satisfy  $(yCC_d)$  for some  $d$ , then we can re-assign  $y = 1$  to get another optimal assignment  $\sigma'_2$  that satisfies all  $i$  clauses in  $(yCC_1) \wedge \dots \wedge (yCC_i)$ . Moreover,  $\sigma'_2$  satisfies at least  $(k - 1) - i$  clauses in the formula  $F$ . Again by the resolution principle,  $\sigma'_2$  plus a proper assignment of the literal  $z$  will satisfy all  $i + 1$  clauses in  $(zC_1) \wedge \dots \wedge (zC_i) \wedge (\bar{z}yC)$ , which thus satisfies at least  $(k - 1) - i + (i + 1) = k$  clauses in  $F \wedge (zC_1) \wedge \dots \wedge (zC_i) \wedge (\bar{z}yC)$ . This shows that  $(F \wedge (zC_1) \wedge \dots \wedge (zC_i) \wedge (\bar{z}yC), k)$  is a Yes-instance.

This completes the proof of the lemma.  $\square$

Our last reduction rule deals with a  $(2, 2)$ -variable, which may not decrease the parameter value  $k$ , but will reduce the number of variables in the formula by eliminating the  $(2, 2)$ -variable. The reduction rule is also based on the resolution principle.

**R-Rule 7.** Let  $z$  be a  $(2, 2)$ -literal in a formula  $F_1 = F \wedge (zy_1C_1) \wedge (zy_2C_2) \wedge (\bar{z}y_3C_3) \wedge (\bar{z}y_4C_4)$ , where each  $y_h$  is an  $(i_h, 1)$ -literal for some  $i_h$ . Then,  $(F \wedge (zy_1C_1) \wedge (zy_2C_2) \wedge (\bar{z}y_3C_3) \wedge (\bar{z}y_4C_4), k) \rightarrow (F \wedge (y_1y_3C_1C_3) \wedge (y_2y_3C_2C_3) \wedge (y_1y_4C_1C_4) \wedge (y_2y_4C_2C_4), k)$ .

**Lemma 3.2** *R-Rule 7 is safe, i.e., R-Rule 7 transforms the instance  $(F_1, k)$  of MAXSAT into an instance  $(F_2, k)$  such that  $(F_1, k)$  is a Yes-instance if and only if  $(F_2, k)$  is a Yes-instance.*

PROOF. Suppose  $(F_1, k)$  is a Yes-instance with an optimal assignment  $\sigma_1$  that satisfies at least  $k$  clauses in  $F_1 = F \wedge (zy_1C_1) \wedge (zy_2C_2) \wedge (\bar{z}y_3C_3) \wedge (\bar{z}y_4C_4)$ . By symmetry, we can assume  $\sigma_1(z) = 0$ .

If  $(zy_1C_1)$  is not satisfied by  $\sigma_1$ , then  $\sigma_1(y_1) = 0$ . Now by re-assigning  $y_1 = 1$ , the unsatisfied clause  $(zy_1C_1)$  becomes satisfied and, because  $y_1$  is an  $(i_1, 1)$ -literal, only one clause that is satisfied by  $\sigma_1$  (i.e., the clause that contains  $\bar{y}_1$ ) may become unsatisfied. Therefore, the re-assignment  $y_1 = 1$  converts  $\sigma_1$  into another optimal assignment  $\sigma'_1$ , with  $\sigma'_1(z) = 0$  and  $\sigma'_1(y_1) = 1$ . Now the optimal assignment  $\sigma'_1$  satisfies at least three clauses in  $(zy_1C_1) \wedge (zy_2C_2) \wedge (\bar{z}y_3C_3) \wedge (\bar{z}y_4C_4)$ . If  $(zy_2C_2)$  is still not satisfied by  $\sigma'_1$ , then we know that  $y_1 \neq y_2$ , so we similarly re-assign the  $(i_2, 1)$ -literal  $y_2$  by  $y_2 = 1$ , which will give us a third optimal assignment  $\sigma''_1$  that satisfies all four clauses in  $(zy_1C_1) \wedge (zy_2C_2) \wedge (\bar{z}y_3C_3) \wedge (\bar{z}y_4C_4)$  (note that re-assigning  $y_2 = 1$  cannot make  $(zy_1C_1)$  become unsatisfied: by R-Rule 3,  $y_2 \neq \bar{y}_1$ ). Moreover,  $\sigma''_1$  satisfies at least  $k - 4$  clauses in  $F$ . By the resolution principle,  $\sigma''_1$  also satisfies all four clauses in  $(y_1y_3C_1C_3) \wedge (y_2y_3C_2C_3) \wedge (y_1y_4C_1C_4) \wedge (y_2y_4C_2C_4)$ . Thus,  $\sigma''_1$  satisfies at least  $k$  clauses in  $F \wedge (y_1y_3C_1C_3) \wedge (y_2y_3C_2C_3) \wedge (y_1y_4C_1C_4) \wedge (y_2y_4C_2C_4)$ , i.e.,  $(F \wedge (y_1y_3C_1C_3) \wedge (y_2y_3C_2C_3) \wedge (y_1y_4C_1C_4) \wedge (y_2y_4C_2C_4), k)$  is a Yes-instance.

For the other direction, suppose that  $(F \wedge (y_1y_3C_1C_3) \wedge (y_2y_3C_2C_3) \wedge (y_1y_4C_1C_4) \wedge (y_2y_4C_2C_4), k)$  is a Yes-instance with an optimal assignment  $\sigma_2$  that satisfies at least  $k$  clauses in  $F \wedge (y_1y_3C_1C_3) \wedge (y_2y_3C_2C_3) \wedge (y_1y_4C_1C_4) \wedge (y_2y_4C_2C_4)$ . Again if  $\sigma_2$  does not satisfy, say  $(y_1y_3C_1C_3)$ , then we can re-assign  $y_1 = 1$ , which does not decrease the number of satisfied clauses, thus gives another optimal assignment  $\sigma'_2$  that satisfies the clauses  $(y_1y_3C_1C_3)$  and  $(y_1y_4C_1C_4)$ . If  $(y_2y_3C_2C_3)$  or  $(y_2y_4C_2C_4)$  is still unsatisfied by  $\sigma'_2$ , then we re-assign  $y_2 = 1$  to get a third optimal assignment  $\sigma''_2$  that satisfies all four clauses in  $(y_1y_3C_1C_3) \wedge (y_2y_3C_2C_3) \wedge (y_1y_4C_1C_4) \wedge (y_2y_4C_2C_4)$  (again note that by R-Rule 3,  $y_2$  cannot be  $\bar{y}_1$ ). Now using the resolution principle,  $\sigma''_2$  plus a proper assignment to  $z$  will satisfy all four clauses in  $(zy_1C_1) \wedge (zy_2C_2) \wedge (\bar{z}y_3C_3) \wedge (\bar{z}y_4C_4)$ , so will satisfy at least  $k$  clauses in  $F \wedge (zy_1C_1) \wedge (zy_2C_2) \wedge (\bar{z}y_3C_3) \wedge (\bar{z}y_4C_4)$ . In conclusion,  $(F \wedge (zy_1C_1) \wedge (zy_2C_2) \wedge (\bar{z}y_3C_3) \wedge (\bar{z}y_4C_4), k)$  is a Yes-instance.

This completes the proof of the lemma.  $\square$

We remark that although R-Rule 7 decreases the number of variables in the formula, it may significantly increase the size of the formula. This, however, does not diminish the usability of the rule: the rule does not increase the parameter value  $k$ . Thus, by the kernelization algorithm for MAXSAT [5], once the size of the formula  $F$  in an instance  $(F, k)$  gets too large, we can always apply a polynomial-time process to reduce the formula size and bound it by  $O(k^2)$ .

## 4 Branching rules

If any of R-Rules 1-7 is applicable on a formula  $F$ , we apply the rule, which either decreases the parameter value  $k$  (R-Rules 1-6) or reduces the number of variables without increasing the parameter value (R-Rule 7). A formula  $F$  is *irreducible* if none of R-Rules 1-7 is applicable on  $F$ . It is obvious that each of R-Rules 1-7 takes polynomial time, and that these rules can be applied at most polynomial many times (this holds true for R-Rule 7 because the MAXSAT problem has a kernel of size  $O(k^2)$  [5]). Thus, with a polynomial-time preprocessing, we can always reduce a

given instance into an irreducible instance. Therefore, without loss of generality, we can assume that the formula  $F$  in our discussion is always irreducible.

In this section, we present a series of *branching rules* (B-Rules), which on an instance  $(F, k)$  constructs a collection of new instances such that  $(F, k)$  is a Yes-instance if and only if at least one of the new instances is a Yes-instance. Again, we assume that the B-Rules are applied in order so that B-Rule  $j$  is applied only when none of B-Rules  $i$  with  $i < j$  is applicable.

For a given instance  $(F, k)$ , and an  $(i, j)$ -literal  $z$  in  $F$ , we say that we “branch on  $z$ ” if we construct two instances  $(F_{z=1}, k - i)$  and  $(F_{z=0}, k - j)$ , and recursively work on the instances, where  $F_{z=1}$  and  $F_{z=0}$  are the formula  $F$  with the assignments  $z = 1$  and  $z = 0$ , respectively.

As well known, branching on a high degree variable has a sufficiently good branching complexity.

**Lemma 4.1 (B-Rule 1)** *If an irreducible formula  $F$  contains a  $6^+$ -variable  $x$  or a  $(3, 2)$ -literal  $x$ , then branch on  $x$ . The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. By R-Rule 2, there is no pure literal in the formula  $F$ . As noted before, a less balanced branching has a higher branching complexity. Thus, branching on a  $6^+$ -variable is not inferior to the  $(5, 1)$ -branching. Also, branching on a  $(3, 2)$ -literal is not inferior to the  $(3, 2)$ -branching. Since  $\rho(5, 1) = \rho(3, 2) (\approx 1.3248)$ , branching on  $x$  is not inferior to the  $(3, 2)$ -branching.  $\square$

We also note Bliznets and Golovnev’s result for branching on 3-variables [4].

**Lemma 4.2 ([4]) (B-Rule 2)** *If an irreducible formula  $F$  contains a 3-variable, then we can make a branching that is not inferior to the  $(6, 1)$ -branching, thus is not inferior to the  $(3, 2)$ -branching.*

PROOF. It is proved in [4] that when there is a 3-variable in the formula, then we can always make a branching whose branching vector is either  $(6, 1)$ , or  $(4, 2)$ , or  $(3, 3)$ . Since  $\rho(6, 1) \approx 1.2852$ ,  $\rho(4, 2) \approx 1.2721$ , and  $\rho(3, 3) \approx 1.2600$ , the branching is not inferior to the  $(6, 1)$ -branching. Since  $\rho((3, 2)) \approx 1.3248$ , the branching is not inferior to the  $(3, 2)$ -branching.  $\square$

With Lemmas 4.1-4.2, we can assume in the following that a given irreducible formula  $F$  contains only  $(4, 1)$ -,  $(3, 1)$ -, and  $(2, 2)$ -literals and their negations.

**Lemma 4.3 (B-Rule 3)** *Suppose that  $z$  be an  $(i, 1)$ -literal in an irreducible formula  $F$  such that  $(\bar{z}y_1 \cdots y_h)$  is not a unit clause. Then branch with (B1)  $z = 1$ ; and (B2)  $z = y_1 = \cdots = y_h = 0$ . The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. We first verify the correctness of the branch. If there is an optimal assignment  $\sigma$  for  $F$  such that  $\sigma(z) = 0$  but  $\sigma(y_b) = 1$  for some  $b$ ,  $1 \leq b \leq h$ . Then we can simply change the value of  $z$  from 0 to 1. This change does not decrease the number of satisfied clauses since  $(\bar{z}y_1 \cdots y_h)$  is the only clause containing  $\bar{z}$  while  $y_b$  has value 1. Therefore, in this case, we also have an optimal assignment to  $F$  that assigns value 1 to  $z$ . As a consequence, if no optimal assignment assigns  $z$  value 1, then every optimal assignment must assign value 0 to all literals  $z, y_1, \dots, y_h$ .

By the assumption,  $i \geq 3$ . Therefore, branch (B1) with  $z = 1$  satisfies at least 3 clauses. On the other hand, because of R-Rule 6,  $y_1$  cannot be an  $(j, 1)$ -literal for any  $j$ , so branch (B2) that assigns  $y_1 = 0$  satisfies at least 2 clauses. As a result, the branch is not inferior to the  $(3, 2)$ -branching.  $\square$

An  $(i, 1)$ -literal  $z$  in a formula  $F$  is an  $(i, 1)$ -*singleton* if the clause containing  $\bar{z}$  is a unit clause. With Lemma 4.3, we can assume in the following that all  $(i, 1)$ -literals are  $(i, 1)$ -singletons. A literal is a *singleton* if it is an  $(i, 1)$ -singleton for some  $i$ .

**Lemma 4.4 (B-Rule 4)** *Let  $z$  be an  $(i, 1)$ -literal in an irreducible formula  $F$  that contains a 2-clause  $(zy)$ . Then branch with: (B1)  $z = 1$ ; and (B2)  $z = 0$  and  $y = 1$ . The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. If there is an optimal assignment  $\sigma$  for  $F$  with  $\sigma(z) = 0$  and  $\sigma(y) = 0$ . Then change the value of  $z$  from 0 to 1. This change satisfies the clause  $(zy)$  that is not satisfied by  $\sigma$ . Moreover, this change can make at most one clause satisfied by  $\sigma$  to become unsatisfied (i.e., the clause containing  $\bar{z}$ ). Therefore, the new assignment is still an optimal assignment for  $F$  but it assigns  $z = 1$ . As a consequence, if the formula  $F$  has no optimal assignment that assigns  $z = 1$ , then each optimal assignment must assign  $z = 0$  and  $y = 1$ . This verifies the correctness of the branching.

By the assumption,  $i \geq 3$ . Thus, branch (B1) with  $z = 1$  satisfies at least 3 clauses. On the other hand,  $z = 0$  and  $y = 1$  satisfy at least two clauses: the one containing  $\bar{z}$  and the clause  $(zy)$ . Thus, branch (B2) satisfies at least 2 clauses. As a result, the branching is not inferior to the  $(3, 2)$ -branching.  $\square$

With Lemma 4.4 and because of R-Rule 2, every  $(i, 1)$ -literal is contained in a  $3^+$ -clause.

The next nine branching rules are dealing with  $(2, 2)$ -literals, which present the most difficult cases for our algorithm.

**Lemma 4.5 (B-Rule 5)** *If there is a  $(2, 2)$ -literal  $z$  with two clauses  $(zy_1C_1)$  and  $(zy_2C_2)$  in the irreducible formula  $F$ , where  $y_1$  and  $y_2$  are literals of the same 4-variable  $y$ , then branch on  $z$  and apply R-Rule 2 or 3. The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. The branch  $z = 0$  satisfies 2 clauses. The branch with  $z = 1$  also satisfies 2 clauses and leaves  $y$  as a 2-variable, which, by R-Rule 2 or 3, can further reduce the parameter value  $k$  by at least 1. Therefore, the branching is not inferior to the  $(3, 2)$ -branching.  $\square$

**Lemma 4.6 (B-Rule 6)** *If two clauses both contain literals  $z$  and  $y$ , where  $z$  is a  $(2, 2)$ -literal, then branch with: (B1)  $y = 0$ ; and (B2)  $y = 1$  followed by an application of R-Rule 2. The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. Since we assume that B-Rule 5 is not applicable,  $y$  must be a  $(4, 1)$ -literal. The branch (B1) satisfies 1 clause. Now consider the branch (B2). Since R-Rule 7 is not applicable, at least one of the two clauses containing  $\bar{z}$  does not contain  $y$ . Therefore, assigning  $y = 1$  satisfies 4 clauses and leaves  $\bar{z}$  as a pure literal, for which R-Rule 2 can further decrease the parameter value by at least 1. In conclusion, the branch (B2) satisfies at least 5 clauses. Since  $\rho((5, 1)) = \rho((3, 2))$ , the branching is not inferior to the  $(3, 2)$ -branching.  $\square$

**Lemma 4.7 (B-Rule 7)** *If there is a  $(2, 2)$ -literal  $z$  with two clauses  $(zC_1)$  and  $(zC_2)$  in the irreducible formula  $F$  such that  $(\bar{z})$  is a unit clause, then branch with: (B1)  $z = 1$ ,  $C_1C_2 = 0$ ; and (B2)  $z = 0$ . The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. We first consider the correctness of B-Rule 7. Suppose that an optimal assignment  $\sigma$  assigns  $z = 1$  and  $C_1 = 1$ . Then we can reassign  $z = 0$ . This makes the unsatisfied clause  $(\bar{z})$  become satisfied, while can make at most one clause (i.e., the clause  $(zC_2)$ ) satisfied by  $\sigma$  become unsatisfied. Therefore, the resulting assignment is also an optimal assignment. By symmetry, the case with  $z = 1$  and  $C_2 = 1$  can be dealt with by the same argument. Therefore, if an optimal assignment assigns  $z = 1$ , then we can always derive that there is an optimal assignment that assigns  $z = 1$  and  $C_1C_2 = 0$  or assigns  $z = 0$ . This proves the correctness of B-Rule 7.



The branch with  $z = 0$  satisfies 2 clauses. Since R-Rule 2 is not applicable, at least one of  $C_1$  and  $C_2$  is not empty. By B-Rule 6,  $C_1$  and  $C_2$  share no common literals. Therefore, assigning  $C_1 C_2 = 0$  satisfies at least one clause not containing  $z$ . Thus, assigning  $z = 1$ ,  $C_1 C_2 = 0$  satisfies at least 3 clauses. This shows that B-Rule 7 is not inferior to the  $(3, 2)$ -branching.  $\square$

By Lemma 4.7, if B-Rule 7 is not applicable, then  $(2, 2)$ -literals can only be in  $2^+$ -clauses.

**Lemma 4.8 (B-Rule 8)** *If for the two clauses containing a  $(2, 2)$ -literal  $z$ , one contains a  $(i, 1)$ -literal  $y_1$  and the other contains a  $(2, 2)$ -literal  $y_2$ , then branch with: (B1)  $y_2 = 1$ , then apply R-Rule 6; and (B2)  $y_2 = 0$ . The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. Suppose that the two clauses containing the literal  $z$  are  $(zy_1 C_1)$  and  $(zy_2 C_2)$ , where  $y_2$  cannot be  $\bar{z}$  because R-Rule 1 is not applicable. Moreover, neither of  $y_2$  and  $\bar{y}_2$  is in  $(zy_1 C_1)$  since B-Rule 5 is not applicable. Therefore, after assigning  $y_2 = 1$ ,  $\bar{z}$  becomes an  $(j, 1)$ -literal (where  $j$  could be smaller than 2), and the clause  $(zy_1 C_1)$  contains  $z = \bar{z}$  and the  $(i, 1)$ -literal  $y_1$ , on which R-Rule 6 is applicable to further reduce the parameter value by 1. Therefore, assigning  $y_2 = 1$  plus applying R-Rule 6 will reduce the parameter value by 3. On the other hand, assigning  $y_2 = 0$  satisfies 2 clauses. In conclusion, the branching is not inferior to the  $(3, 2)$ -branching.  $\square$

By Lemma 4.8, if B-Rule 8 is not applicable, then for any  $(2, 2)$ -literal  $z$ , the two clauses containing  $z$  cannot have one containing a singleton and the other containing a  $(2, 2)$ -literal other than  $z$ . Therefore, the two clauses containing  $z$  should either contain only singletons or contain only  $(2, 2)$ -literals. Suppose that the four clauses containing either  $z$  or  $\bar{z}$  are  $(zC_1)$ ,  $(zC_2)$ ,  $(\bar{z}D_1)$ , and  $(\bar{z}D_2)$ . Since B-Rule 7 is not applicable, none of  $C_1$ ,  $C_2$ ,  $D_1$ , and  $D_2$  can be empty. Moreover, either all literals in  $C_1 C_2$  are singletons, or all literals in  $C_1 C_2$  are  $(2, 2)$ -literals. The same argument also applies for  $D_1 D_2$ . Moreover, since R-Rule 7 is not applicable, not all literals in  $C_1 C_2 D_1 D_2$  can be singletons. In summary, we must have one the following two cases: (1) all literals in  $C_1 C_2 D_1 D_2$  are  $(2, 2)$ -literals; and (2) one of  $C_1 C_2$  and  $D_1 D_2$  contains only singletons and the other contains only  $(2, 2)$ -literals. We introduce two terminologies for the  $(2, 2)$ -literals in these two different situations.

**Definition** A  $(2, 2)$ -literal  $z$  is *skewed* if for  $z_1$ , which is either  $z$  or  $\bar{z}$ , all other literals in the two clauses containing  $z_1$  are singletons and all literals in the two clauses containing  $\bar{z}_1$  are  $(2, 2)$ -literals. A  $(2, 2)$ -literal  $z$  is *evened* if the four clauses containing either  $z$  or  $\bar{z}$  contain only  $(2, 2)$ -literals.

Thus, if none of B-Rules 1-8 is applicable, then an irreducible formula  $F$  contains only  $(3, 1)$ -singletons,  $(4, 1)$ -singletons, skewed  $(2, 2)$ -literals, and evened  $(2, 2)$ -literals.

**Lemma 4.9 (B-Rule 9)** *If an evened  $(2, 2)$ -literal  $z$  is in a 2-clause, then pick any literal  $y \neq \bar{z}$  in a clause containing  $\bar{z}$ , and branch on  $y$ . The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. Let the 2-clause containing  $z$  be  $(zz_1)$ , and let  $(\bar{z}yC_1)$  be a clause containing  $\bar{z}$ . Since B-Rule 7 and R-Rule 1 are not applicable, the literal  $y$  must exist and  $y \neq z, \bar{z}$ . Moreover, since  $z$  is an evened  $(2, 2)$ -literal,  $y$  is a  $(2, 2)$ -literal. Because B-Rule 5 is not applicable, the other clause containing  $y$  contains neither  $z$  nor  $\bar{z}$ . Therefore, Assigning  $y = 1$  will make  $z$  a  $(2, 1)$ -literal. Now either R-Rule 2 (in case  $z_1 = \bar{y}$ ) or R-Rule 4 (in case  $z_1 \neq \bar{y}$ ) will become applicable, which will further reduce the parameter value  $k$  by 1. In summary, assigning  $y = 1$  plus a reduction rule will decrease the parameter value by at least 3. For the other direction, assigning  $y = 0$  decreases the parameter value  $k$  by 2. In conclusion, the branching is not inferior to the  $(3, 2)$ -branching.  $\square$

By Lemma 4.9, if B-Rule 9 is not applicable, then every  $(2,2)$ -literal in a 2-clause is skewed. This combined with the fact that B-Rule 4 is not applicable guarantees that every literal in a 2-clause is a skewed  $(2,2)$ -literal. The next branching rule is to deal with literals in 2-clauses.

**Lemma 4.10 (B-Rule 10)** *For a given 2-clause  $(zy)$ , let the two clauses containing the literal  $\bar{z}$  be  $(\bar{z}C_1)$  and  $(\bar{z}C_2)$ . Branch with: (B1)  $y = 1$ ; (B2)  $y = 0, z = 1$ ; and (B3)  $y = z = C_1 = C_2 = 0$ . The branching is not inferior to the  $(3,2)$ -branching.*

PROOF. Since B-Rule 9 and B-Rule 4 are not applicable, both  $z$  and  $y$  are skewed  $(2,2)$ -literals. We first consider the correctness of the branching rule B-Rule 10. Suppose that there is an optimal assignment  $\sigma$  that assigns  $y = z = 0$  but  $C_1 = 1$ . We can change the assignment  $z = 0$  to  $z = 1$ . This reassignment makes the 2-clause  $(zy)$  unsatisfied by  $\sigma$  become satisfied, and can change at most one clause, i.e., the clause  $(\bar{z}C_2)$  from being satisfied to being unsatisfied. Therefore, the new assignment is also an optimal assignment that is covered by the branch (B2). The same argument applies for the case  $y = z = 0$  but  $C_2 = 1$ . Therefore, if no optimal assignment is covered by the branches (B1) and (B2), then optimal assignments must assign  $y = z = C_1 = C_2 = 0$ , which is covered by the branch (B3). This verifies the correctness of the branching rule B-Rule 10.

The branch (B1) satisfies 2 clauses. Because of B-Rule 5 and the clause  $(zy)$ , a clause containing  $\bar{y}$  cannot contain  $z$ . Therefore, assigning  $y = 0$  will satisfy two clauses that do not contain  $z$ , which derives that the branch (B2) with  $y = 0$  and  $z = 1$  will satisfy 4 clauses. Finally, consider the branch (B3). Because of B-Rule 7, neither  $C_1$  nor  $C_2$  can be empty. Moreover, since  $z$  is a skewed  $(2,2)$ -literal and  $y$  is a  $(2,2)$ -literal, all literals in  $C_1$  and  $C_2$  are singletons. By B-Rule 4, singletons can only be contained in  $3^+$ -clauses. Therefore, we can assume that  $C_1 = (y_1y'_1C'_1)$  and  $C_2 = (y_2y'_2C'_2)$ , where  $y_1, y'_1, y_2, y'_2$  are all singletons. Because of B-Rule 6,  $y_1, y'_1, y_2, y'_2$  are four distinct singletons. Thus, the branch (B3) satisfies 2 clauses by  $y = 0$ , another 2 clauses by  $z = 0$  (note that  $C_1$  and  $C_2$  contain only singletons so cannot contain  $\bar{y}$ ), and at least another 4 clauses by  $C_1 = C_2 = 0$  (note that each clause containing the negation of  $y_1, y'_1, y_2, y'_2$  is a unit clause). In summary, the branch (B3) satisfies at least 8 clauses. Therefore, the branching is not inferior to the  $(8,4,2)$ -branching. Since  $\rho(8,4,2) \approx 1.3248 = \rho(3,2)$ , the branching is not inferior to the  $(3,2)$ -branching.  $\square$

By Lemma 4.10, if B-Rule 10 is not applicable, then all  $2^+$ -clauses are  $3^+$ -clauses.

**Lemma 4.11 (B-Rule 11)** *If a clause  $(zyC_1)$  contains two  $(2,2)$ -literals  $z$  and  $y$ , where  $y$  is a skewed  $(2,2)$ -literal and the other clause containing  $z$  is  $(zC_2)$ , then branch with: (B1)  $z = 0$ ; (B2)  $z = 1, yC_1 = 0$ ; and (B3)  $z = 1, C_2 = 0$ . The branching is not inferior to the  $(3,2)$ -branching.*

PROOF. We first consider the correctness of B-Rule 11. Suppose that an optimal assignment  $\sigma$  assigns  $yC_1 = 1$  and  $C_2 = 1$ , then we can assume  $\sigma$  also assigns  $z = 0$  because assigning  $z = 1$  would not increase the number of satisfied clauses. Therefore, if no optimal assignment assigns  $z = 0$ , then an optimal assignment must either assign  $z = 1, yC_1 = 0$  or assign  $z = 1, C_2 = 0$ , which are covered by the branches (B2) and (B3), respectively.

Let the four clauses containing either  $z$  or  $\bar{z}$  be  $(zyC_1)$ ,  $(zC_2)$ ,  $(\bar{z}C_3)$ , and  $(\bar{z}C_4)$ . Because B-Rule 10 is not applicable, all these clauses are  $3^+$ -clauses. Thus, we can assume  $C_1 = (y_1C'_1)$  and  $C_2 = (y_2y'_2C'_2)$ . Since  $y$  is a  $(2,2)$ -literal and B-Rule 8 is not applicable, the three literals  $y_1, y_2$ , and  $y'_2$  are all  $(2,2)$ -literals.

The branch (B1) with  $z = 0$  satisfies 2 clauses. For the branch (B2), because B-Rule 5 and R-Rule 1 are not applicable,  $\bar{y}$  cannot be in the clauses  $(zyC_1)$  and  $(zC_2)$ . Therefore, assigning  $z = 1$  and  $y = 0$  satisfies 4 clauses. Also, because R-Rule 1 and B-Rule 5 are not applicable,  $\bar{y}_1$

cannot be in the clauses  $(zyC_1) = (zyy_1C'_1)$  and  $(zC_2)$ . Moreover,  $\bar{y}_1$  and  $\bar{y}$  cannot be in the same clause:  $y$  is a skewed  $(2, 2)$ -literal and the clause  $(zyC_1)$  also contains the  $(2, 2)$ -literal  $z$ . Thus, all literals contained in a clause containing  $\bar{y}$ , except  $\bar{y}$ , are singletons. while  $\bar{y}_1$  is a  $(2, 2)$ -literal. Therefore, besides the 4 clauses satisfied by  $z = 1$  and  $y = 0$ , assigning  $y_1 = 0$  satisfies 2 additional clauses. This shows that the branch (B2) with  $z = 1$ ,  $yC_1 = yy_1C'_1 = 0$  satisfies at least 6 clauses. Finally, we consider the branch (B3). Because B-Rule 5 and R-Rule 1 are not applicable, neither  $\bar{y}_2$  nor  $\bar{y}'_2$  can be in the clauses  $(zyC_1)$  and  $(zC_2) = (zy_2y'_2C'_2)$ . Moreover, because B-Rule 6 is not applicable,  $\bar{y}_2$  and  $\bar{y}'_2$  cannot be contained in only two clauses. Thus, there are at least three clauses that contain either  $\bar{y}_2$  or  $\bar{y}'_2$  (or both). Therefore, besides the 2 clauses satisfied by  $z = 1$ , assigning  $C_2 = y_2y'_2C'_2 = 0$  satisfies at least 3 additional clauses. This derives that the branch (B3) satisfies at least 5 clauses. In summary, the branching rule B-Rule 11 is not inferior to the  $(6, 5, 2)$ -branching. Since  $\rho((6, 5, 2)) = \rho((3, 2))$ , the branching B-Rule 11 is not inferior to the  $(3, 2)$ -branching.  $\square$

Let  $y$  be a skewed  $(2, 2)$ -literal. By Lemma 4.11, if B-Rule 11 is not applicable, a clause  $C$  containing  $y$  cannot contain other  $(2, 2)$ -literals. Therefore, all other literals in the clause  $C$  are singletons. Note that  $\bar{y}$  is also a skewed  $(2, 2)$ -literal, so all other literals in a clause containing  $\bar{y}$  are also singletons. However, in this case, R-Rule 7 would have become applicable. Therefore, if B-Rule 11 is not applicable, then an irreducible formula  $F$  contains no skewed  $(2, 2)$ -literals. In conclusion, the formula  $F$  contains only  $(4, 1)$ -singletons,  $(3, 1)$ -singletons, and evened  $(2, 2)$ -literals, and all clauses in the formula  $F$  that are not unit are  $3^+$ -clauses.

**Lemma 4.12 (B-Rule 12)** *If the clauses containing a  $(2, 2)$ -literal  $z$  are  $(zy_1C_1)$  and  $(zy_2C_2)$ , and there is a third clause  $(y_1\bar{y}_2C_3)$ , then branch on  $z$  and in the branch  $z = 1$  also apply R-Rule 6. The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. Branching with  $z = 0$  satisfies 2 clauses. Since B-Rule 11 is not applicable and  $z$  is an evened  $(2, 2)$ -literal, both  $y_1$  and  $y_2$  are  $(2, 2)$ -literals. Moreover, by B-Rule 6,  $y_1$  is not in  $(zy_2C_2)$ , and  $y_2$  is not in  $(zy_1C_1)$ . Thus, assigning  $z = 1$  satisfies 2 clauses and also makes both  $\bar{y}_1$  and  $\bar{y}_2$  become singletons. Now R-Rule 6 can be applied on the clause  $(y_1\bar{y}_2C_3)$  and further decreases the parameter value  $k$  by 1. In conclusion, assigning  $z = 1$  plus applying R-Rule 6 will decrease the parameter value by 3. Therefore, the branching is not inferior to the  $(3, 2)$ -branching.  $\square$

With Lemma 4.12, we are ready to eliminate all  $(2, 2)$ -literals.

**Lemma 4.13 (B-Rule 13)** *For clauses  $(zy_1C_1)$ ,  $(\bar{z}y_2C_2)$ ,  $(y_1D_1)$ , and  $(y_2D_2)$ , where  $z$  is a  $(2, 2)$ -literal and  $(y_1D_1)$  could be  $(y_2D_2)$ , branch with: (B1)  $z = 1$ ,  $y_1 = 0$ , then apply B-Rule 2; (B2)  $z = y_1 = 1$ ,  $D_1 = 0$ ; (B3)  $z = 0$ ,  $y_2 = 0$ , then apply B-Rule 2; and (B4)  $z = 0$ ,  $y_2 = 1$ ,  $D_2 = 0$ . The branching is not inferior to the  $(10, 10, 6, 6, 5, 5)$ -branching, which is not inferior to the  $(3, 2)$ -branching.*

PROOF. We first verify the correctness of the branching rule B-Rule 13. Since  $z$  is an evened  $(2, 2)$ -literal,  $y_1$  is a  $(2, 2)$ -literal. Suppose that there is an optimal assignment  $\sigma$  that assigns  $z = 1$ . If  $\sigma$  also assigns  $y_1 = 1$  and  $y'_1 = 1$  for some literal  $y'_1$  in  $D_1$ , then we can change the value of  $y_1$  from 1 to 0. This does not decrease the number of satisfied clauses since the clauses  $(zy_1C_1)$  and  $(y_1D_1)$  containing  $y_1$  remain satisfied. Therefore, under the assumption that there are optimal assignments that assign  $z = 1$ , if none of these assignments assigns  $y_1 = 0$ , then such optimal assignments must assign  $y_1 = 1$  and  $D_1 = 0$ . This verifies the correctness of the branches (B1)-(B2) for the situation where there is an optimal assignment with  $z = 1$ . Since  $\bar{z}$  is also a  $(2, 2)$ -literal, by symmetry,

the correctness of the branches (B3)-(B4), which is for the situation where there is an optimal assignment with  $z = 0$ , follows.

Since B-Rule 10 is not applicable, we can assume  $(y_1 D_1) = (y_1 y'_1 y''_1 D'_1)$ , where  $y'_1$  and  $y''_1$  are two different  $(2, 2)$ -literals. Let the two clauses containing  $z$  be  $(z y_1 C_1)$  and  $(z C'_1)$ . We have the following observations:

1. the clause  $(z y_1 C_1)$  contains neither  $\bar{y}_1$  (by R-Rule 1), nor any of  $y'_1, y''_1, \bar{y}'_1, \bar{y}''_1$  (because of B-Rule 5 and the clause  $(y_1 D_1) = (y_1 y'_1 y''_1 D'_1)$ );
2. the clause  $(z C'_1)$  contains neither of  $y_1$  and  $\bar{y}_1$  (because of B-Rule 5 and the clause  $(z y_1 C_1)$ ), nor any of  $\bar{y}'_1$  and  $\bar{y}''_1$  (because of B-Rule 12 and the clauses  $(z y_1 C_1)$  and  $(y_1 y'_1 y''_1 D'_1)$ ); and
3. the clause  $(y_1 D_1) = (y_1 y'_1 y''_1 D'_1)$  contains neither of  $z$  and  $\bar{z}$  (because of B-Rule 5 and the clause  $(z y_1 C_1)$ ) nor any of  $\bar{y}_1, \bar{y}'_1, \bar{y}''_1$  (because of R-Rule 1).

Thus, there are four different clauses  $(z y_1 C_1)$ ,  $(z C'_1)$ ,  $(\bar{y}_1 C_2)$ , and  $(\bar{y}_1 C'_2)$ , which are satisfied by the assignment  $z = 1$  and  $y = 0$ . By B-Rule 10,  $C_1$  contains at least one more  $(2, 2)$ -literal  $z_1$ . By B-Rule 5, neither  $z_1$  nor  $\bar{z}_1$  is in  $C'_1$ . Let  $x_1$  be the variable for the literal  $z_1$ , then after the assignment  $z = 1$  and  $y = 0$ , the variable  $x_1$  will become an  $h$ -variable, where  $1 \leq h \leq 3$ . Thus, B-Rule 2 is applicable on  $x_1$ , which, by Lemma 4.2, is not inferior to the  $(6, 1)$ -branching. Thus, the branch (B1) first reduces the parameter  $k$  by 4, then makes a branching not inferior to the  $(6, 1)$ -branching. Combining these two steps, the branch (B1) can be regarded as a branching not inferior to the  $(4 + 6, 4 + 1) = (10, 5)$ -branching.

For the branch (B2),  $z = 1$  satisfies 2 clauses  $(z y_1 C_1)$  and  $(z C'_1)$ , and  $y_1 = 1$  satisfies 1 additional clause  $(y_1 D_1)$ . By 1-3 listed above, none of  $(z y_1 C_1)$ ,  $(z C'_1)$ ,  $(y_1 D_1)$  contains any of  $\bar{y}'_1, \bar{y}''_1$ . Moreover, since both  $y'_1$  and  $y''_1$  are  $(2, 2)$ -literals and by B-Rule 6, there are at least 3 clauses containing either  $\bar{y}'_1$  or  $\bar{y}''_1$  (or both). Therefore, assigning  $D_1 = y'_1 y''_1 D'_1 = 0$  satisfies at least 3 additional clauses. In total, the branch (B2) satisfies at least 6 clauses.

By symmetry and a completely similar analysis, we can show that the branch (B3) is equivalent to a further branching not inferior to the  $(10, 5)$ -branching, and that the branch (B4) satisfies at least 6 clauses. In conclusion, the branching rule B-Rule 13 is not inferior to the  $(10, 5, 6, 10, 5, 6)$ -branching. Since  $\rho(10, 5, 6, 10, 5, 6) \approx 1.3204$ , while  $\rho(3, 2) \approx 1.3248$ , we conclude that the branching rule B-Rule 13 is not inferior to the  $(3, 2)$ -branching.  $\square$

By Lemma 4.13, if the branching rule B-Rule 13 is not applicable, then there will be no  $(2, 2)$ -literals. Thus, all literals in an irreducible formula  $F$  are either  $(3, 1)$ -singletons or  $(4, 1)$ -singletons, or their negations. Moreover, all clauses that are not unit clauses are  $3^+$ -clauses. The following branching rule will further eliminate all 3-clauses.

**Lemma 4.14 (B-Rule 14)** *If there is a 3-clause  $(z_1 z_2 z_3)$ , then branch with: (B1)  $z_1 = 1$ ; (B2)  $z_1 = 0, z_2 = 1$ ; and (B3)  $z_1 = z_2 = 0, z_3 = 1$ . The branching is not inferior to the  $(3, 2)$ -branching.*

PROOF. We first verify the correctness of the branching. Since B-Rule 13 is not applicable, all literals are either  $(4, 1)$ -singletons or  $(3, 1)$ -singletons or their negations. Thus, literals  $z_1, z_2$ , and  $z_3$  in the 3-clause  $(z_1 z_2 z_3)$  are all  $(i, 1)$ -literals, where  $i$  is either 4 or 3. The cases listed in (B1)-(B3) include all cases in which an assignment assigns 1 to at least one of the literals  $z_1, z_2$ , and  $z_3$ . Therefore, we only have to consider the case where an optimal assignment  $\sigma$  assigns 0 to all  $z_1, z_2, z_3$ . In this case,  $\sigma$  does not satisfy the clause  $(z_1 z_2 z_3)$ . Now if we change the assignment  $\sigma$  by assigning 1 instead of 0 to  $z_3$ , then the new assignment  $\sigma'$  satisfies the clause  $(z_1 z_2 z_3)$ , and makes only one clause, i.e.,  $(\bar{z}_3)$  become unsatisfied (note that  $z_3$  is a singleton). Therefore, the

new assignment  $\sigma'$  is also an optimal assignment and its possibility is covered by the branch (B3). This shows that at least one of the branches (B1)-(B3) will lead to an optimal solution.

Since  $z_1$ ,  $z_2$ , and  $z_3$  are all  $(i, 1)$ -literals for  $i$  being either 3 or 4, branch (B1) satisfies at least 3 clauses, branch (B2) satisfies at least 4 clauses: at least 3 clauses by  $z_2 = 1$  and 1 clause by  $z_1 = 0$  (note that since  $z_1$  is an  $(i, 1)$ -singleton, no clause can contain both  $\bar{z}_1$  and  $z_2$ ), and (B3) satisfies at least 5 clauses: at least 3 clauses by  $z_3 = 1$ , 1 clause by  $z_2 = 0$ , and 1 clause by  $z_1 = 0$  (again no clause in  $F$  can contain more than one of  $z_3$ ,  $\bar{z}_2$ ,  $\bar{z}_1$ ). As a result, the branching rule B-Rule 14 is not inferior to the  $(5, 4, 3)$ -branching. The lemma follows since  $\rho(5, 4, 3) = \rho(3, 2) \approx 1.3248$ .  $\square$

Summarizing all Lemmas 4.1-4.14, we conclude that if none of the reduction rules R-Rules 1-7 and the branching rules B-Rules 1-14 is applicable, then all literals are  $(i, 1)$ -singletons or their negations, where  $i$  is either 3 or 4, and all clauses that are not unit clauses are  $4^+$ -clauses.

## 5 An $O^*(1.3248^k)$ -time algorithm for MAXSAT

We are ready to present our main algorithm for the MAXSAT problem.

By Lemmas 4.1-4.14, for any given instance  $(F, k)$  of the MAXSAT problem, we can apply the branching rules B-Rules 1-14, which are not inferior to the  $(3, 2)$ -branching, until the formula  $F$  contains only  $(3, 1)$ -singletons and  $(4, 1)$ -singles, in which all non-unit clauses are  $4^+$ -clauses. Note that in this case, we can assume, without loss of generality, that for each variable  $x$  in  $F$ , the positive literal  $x$  is a singleton while the negative literal  $\bar{x}$  is in a unique unit clause (otherwise we simply exchange  $x$  and  $\bar{x}$ ). An instance  $(F, k)$  will be called a *simplified instance* if every variable in  $F$  is either a 3-singleton or a 4-singleton, and each non-unit clause in  $F$  is a  $4^+$ -clause.

As suggested by Bliznets and Golovnev [4], the MAXSAT problem on simplified instances can be solved by reducing the problem to the MIN SET-COVER problem. An algorithm was presented in [4] that solves the MAXSAT problem on simplified instances in time  $O^*(1.3574^k)$ . We first show how this method can be refined to get an improved algorithm of time  $O^*(1.3226^k)$  for the problem.

### 5.1 Dealing with simplified instances

In the following discussion, we fix a simplified instance  $(F, k)$  for the MAXSAT problem on variables  $\{x_1, x_2, \dots, x_n\}$ , where  $F = C_1 \wedge C_2 \wedge \dots \wedge C_m$ . We first observe the following result, which can be derived based on a general framework proposed by Yannakakis [22]:

**Lemma 5.1** *If  $m + n/2 \geq 1.829k$ , then for the simplified instance  $(F, k)$ , there is an assignment that satisfies at least  $k$  clauses in  $F$ , and the assignment can be constructed in polynomial time.*

PROOF. Since every variable  $x_i$  in  $F$  is a singleton, there are exactly  $n$  unit clauses  $(\bar{x}_i)$ ,  $1 \leq i \leq n$ , and  $m - n$  non-unit clauses. Set  $p = 0.1795$ , and assign each variable  $x_i$  with value 1 with a probability  $p$ . Therefore, each unit clause  $(\bar{x}_i)$  is satisfied with a probability  $1 - p$ . Since each non-unit clause contains at least 4 positive literals, the assignment satisfies a non-unit clause with a probability at least  $1 - (1 - p)^4$ . Therefore, the expected number of satisfied clauses under this random assignment is at least (note  $p = 0.1795$ )

$$\begin{aligned} n(1 - p) + (m - n)(1 - (1 - p)^4) &= n(1 - p) + (m + \frac{n}{2})(1 - (1 - p)^4) - \frac{3n}{2}(1 - (1 - p)^4) \\ &\geq (m + \frac{n}{2})(1 - (1 - p)^4) \geq 1.829k(1 - (1 - p)^4) \geq k. \end{aligned}$$

Now a standard polynomial-time derandomization process, as described in [22], can construct an assignment that satisfies at least  $k$  clauses in the formula  $F$ .  $\square$

Therefore, we only need to consider simplified instances  $(F, k)$  satisfying  $m + n/2 < 1.829k$ . For this kind of instances, we follow the approach proposed in [4] and reduce the problem to the MIN SET-COVER problem. Let  $\mathcal{C}$  be a collection of sets such that  $U = \bigcup_{S \in \mathcal{C}} S$  ( $U$  will be called the *universal set* for  $\mathcal{C}$ ). A subcollection  $\mathcal{C}'$  of  $\mathcal{C}$  is a *set cover* for  $\mathcal{C}$  if  $U = \bigcup_{S \in \mathcal{C}'} S$ . The MIN SET-COVER problem is for a given collection  $\mathcal{C}$  of sets to find a minimum set cover for  $\mathcal{C}$ .

We will denote by  $|S|$  the cardinality of the set  $S$ , i.e., the number of elements in  $S$ . In particular, for a collection  $\mathcal{C}$  of sets,  $|\mathcal{C}|$  is the number of sets in  $\mathcal{C}$ . The following result is due to van Rooij and Bodlaender [19] (see also [4]):

**Theorem 5.2** ([19]) *The MIN SET-COVER problem on instances  $\mathcal{C} = \{S_1, S_2, \dots, S_n\}$ , with  $|S_i| \leq 4$  for all  $S_i$ , can be solved in time  $O^*(1.29^{0.6|U|+0.9|\mathcal{C}|})$ , where  $U$  is the universal set for  $\mathcal{C}$ .*

Now we describe how the simplified instance  $(F, k)$  of the MAXSAT problem is reduced to an instance  $\mathcal{C}_F$  of the MIN SET-COVER problem. Each non-unit clause  $C_h$  in  $F$  corresponds to an element  $a_{C_h}$  in the universal set  $U_F$ , and each variable  $x_i$  in  $F$  corresponds to a set  $S_{x_i}$  in  $\mathcal{C}_F$  such that the set  $S_{x_i}$  contains the element  $a_{C_h}$  if and only if the literal  $x_i$  is in the clause  $C_h$ . Thus, the collection  $\mathcal{C}_F$  consists of  $n$  sets  $S_{x_i}$ ,  $1 \leq i \leq n$ , and the universal set  $U_F$  has  $m - n$  elements.

**Lemma 5.3** *From any minimum set cover  $\mathcal{C}'$  for the collection  $\mathcal{C}_F$ , an optimal assignment to the formula  $F$  in the simplified instance  $(F, k)$  of MAXSAT can be constructed in polynomial time.*

PROOF. As observed in [4], there is an optimal assignment to  $F$  that satisfies all non-unit clauses. In fact, if an optimal assignment  $\sigma$  to  $F$  does not satisfy a non-unit clause  $(x_i C)$ , then we can simply change the value of  $x_i$  from 0 to 1. This will make the clause  $(x_i C)$  satisfied and change only one clause, i.e., the clause  $(\bar{x}_i)$ , from being satisfied to being unsatisfied. Therefore, the resulting assignment is also an optimal assignment to  $F$ . Repeatedly applying this process, we will get an optimal assignment that satisfies all non-unit clauses in  $F$ . Thus, we can assume that an optimal assignment  $\sigma$  satisfies totally  $m - n + q_{\max}$  clauses, including all the  $m - n$  non-unit clauses plus  $q_{\max}$  unit clauses in  $F$ . Let  $T$  be the set of the  $n - q_{\max}$  variables  $x_i$  with  $\sigma(x_i) = 1$ . Then the set  $T$  corresponds to a set cover  $\mathcal{C}_T = \{S_{x_i} \mid x_i \in T\}$  of  $n - q_{\max}$  sets for  $\mathcal{C}_F$ . Let  $t_{\min}$  be the size of a minimum set cover for  $\mathcal{C}_F$ , then  $n - q_{\max} \geq t_{\min}$ .

Let  $\mathcal{C}'$  be a minimum set cover for  $\mathcal{C}_F$ ,  $|\mathcal{C}'| = t_{\min}$ . Then each element  $a_{C_b}$  in the universal set  $U_F$  is in at least one of the sets in  $\mathcal{C}'$ . Equivalently, each non-unit clause  $C_b$  in  $F$  contains at least one variable whose corresponding set is in  $\mathcal{C}'$ . Thus, if we assign value 1 to each of the  $t_{\min}$  variables whose corresponding set is in  $\mathcal{C}'$ , and assign value 0 to each of the rest  $n - t_{\min}$  variables (which will satisfy  $n - t_{\min}$  unit clauses), we get an assignment  $\sigma'$  to the formula  $F$  that satisfies  $(m - n) + (n - t_{\min}) = m - t_{\min}$  clauses. Since an optimal assignment satisfies  $m - n + q_{\max}$  clauses in  $F$ , we have  $m - t_{\min} \leq m - n + q_{\max}$ . Combining this with the inequality in the previous paragraph, we get  $n - q_{\max} = t_{\min}$  so the assignment  $\sigma'$  satisfies  $m - t_{\min} = (m - n) + q_{\max}$  clauses. Thus,  $\sigma'$  is an optimal assignment to  $F$ , and can obviously be constructed from  $\mathcal{C}'$  in polynomial time  $\square$

Now we are ready for a complete algorithm for the MAXSAT problem on simplified instances.

**Theorem 5.4** *The MAXSAT problem on simplified instances can be solved in time  $O^*(1.3226^k)$ .*

PROOF. Let  $(F, k)$  be a simplified instance, where the formula  $F$  consists of  $m$  clauses on  $n$  variables. If  $m + n/2 \geq 1.829k$ , then by Lemma 5.1,  $(F, k)$  is a Yes-instance and we can construct, in polynomial time, an assignment that satisfies at least  $k$  clauses in  $F$ .

Now suppose  $m + n/2 < 1.829k$ . Then we construct the instance  $\mathcal{C}_F$  for the MIN SET-COVER problem, as described above, and apply the algorithm in Theorem 5.2, which constructs a minimum set cover  $\mathcal{C}'$  for  $\mathcal{C}_F$  in time  $O^*(1.29^{0.6|U_F|+0.9|\mathcal{C}_F|}) = O^*(1.29^{0.6(m-n)+0.9n})$ . By Lemma 5.3, from the minimum set cover  $\mathcal{C}'$ , we can construct in polynomial time an optimal assignment  $\sigma$  for  $F$ , from which we can decide whether  $(F, k)$  is a Yes-instance for MAXSAT. The theorem is proved by observing  $0.6(m - n) + 0.9n = 0.6(m + n/2) < 0.6 \cdot 1.829k < 1.098k$  and  $1.29^{1.098k} < 1.3226^k$ .  $\square$

## 5.2 The main algorithm

Summarizing all the discussions, we present our algorithm for the MAXSAT problem in Figure 2.

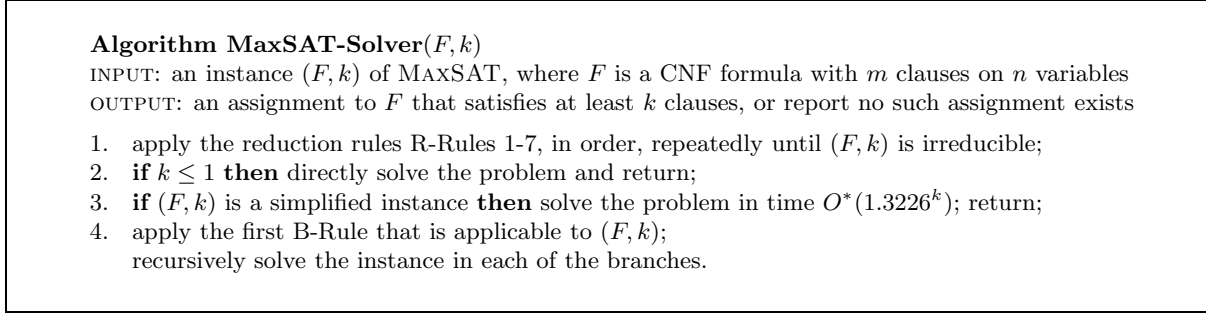


Figure 2: The main algorithm for MAXSAT

**Theorem 5.5** *The algorithm MaxSAT-Solver solves the MAXSAT problem in time  $O^*(1.3248^k)$ .*

PROOF. The execution of the algorithm **Max-SAT-Solver** can be depicted by a search tree  $\mathcal{T}$  in which each node is associated with an instance of the MAXSAT problem. Each leaf of the search tree  $\mathcal{T}$  is associated with either an instance  $(F, k)$  with  $k \leq 1$  for which step 2 of the algorithm directly concludes with a decision, or a simplified instance for which step 3 of the algorithm concludes with a decision. Therefore, by Theorem 5.4, a leaf in the search tree  $\mathcal{T}$  associated with an instance  $(F, k)$  of MAXSAT can be solved in time  $O^*(1.3226^k)$ .

Each internal node of the search tree  $\mathcal{T}$  is associated with an instance  $(F, k)$  and corresponds to an application of one of the branching rules B-Rules 1-14, and its children correspond to the branches of the branching rule. By Lemmas 4.1-4.14, the branching complexity of each of the branching rules B-Rules 1-14 is bounded by 1.3248, which is the branching complexity of the (3, 2)-branching. Now a simple induction shows that the search tree  $\mathcal{T}$ , i.e., the algorithm **Max-SAT-Solver** solves the MAXSAT problem in time  $O^*(1.3248^k)$ .  $\square$

## 6 Conclusion

In this paper we presented an  $O^*(1.3248^k)$ -time algorithm for the MAXSAT problem, which improves the previously best algorithm of time  $O^*(1.358^k)$  for the problem [4]. We showed how the resolution principle can be used effectively in eliminating instance structures that do not support efficient branchings. We also presented techniques to show how the MAXSAT problem on simplified instances can be more effectively reduced to the SET-COVER problem, which leads to a more efficient algorithm for the MAXSAT problem.

The *Exponential Time Hypothesis* [13] implies that there is no  $O^*(2^{o(k)})$ -time algorithm solving the MAXSAT problem. Based on this hypothesis, there is a fixed constant  $c_0 > 1$  such that the MAXSAT problem cannot be solved in time  $O^*(c_0^k)$ . Therefore, there is a limit on the base constant  $c > 1$  for developing improved algorithms of time  $O^*(c^k)$  for the MAXSAT problem. Naturally, it will become more and more difficult to further reduce the value of the constant  $c$ , which perhaps requires more careful and tedious analysis on more and more complicated instance structures. We would like to remark that compared to previous algorithms, our algorithm does not require much more detailed analysis on instance structures. On the other hand, our algorithm reaches the most significant improvement, which improves the base  $c$  by 0.033 over the previous best result [4], while most previous recent works [4, 5] have the improvement bounded by 0.012.

Finally, we would like to point out that further improvement over our algorithm seems to require new techniques and new ideas. Our bound  $O^*(1.3248)$  is “tight” in the sense that all our branching rules, except B-Rules 2 and 13, have their branching complexity equal to 1.3248. In particular, to further improve the bound  $O^*(1.3248)$ , besides handling degree-4 variables more efficiently, we need to deal with (5,1)-literals and (3,2)-literals more efficiently, which introduce more complicated instance structures and have not been considered thoroughly in the literature of the MAXSAT problem.

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